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A NOTE ON BALANCED ROW AND COLUMN DESIGNS

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SUMMARY

The notation $X : YZ$ of Hoblyn *et al.* [2] for classifying row-and-column designs has been modified. Non-existence of some of the balanced row-and-column designs has been established.

Keywords : Row-and-column designs; Classification of designs; Non-existence of designs.

Introduction and Classification of Designs

For classifying row and column designs, Hoblyn *et al.* [2] and Pearce [3] have suggested the notation $X : YZ$ where X, Y and Z respectively indicate the mutual relationship between rows and columns, treatments and rows and treatments and columns. For instance, 0 : 00 is used to indicate designs having all the three classifications pairwise orthogonal; $0 : TT$ indicates a design having row vs. column orthogonal, and treatments vs. rows and treatments vs. columns totally balanced. Here total balance refers to variance balance although it gets extended to efficiency balance in a natural way. We, however, throughout restrict our discussions to variance balance only. Pearce [4] has called X, Y, Z as component designs. Similar classifications are in use for multi-dimensional designs (see, Preece [5]).

The classifications $X : YZ$ is useful as it indicates the structural properties of the incidence matrices of the component designs. However, it suffers from the limitation that it does not always reflect the position in regard to analytical balance of a design. Analytical balance as against the component-wise structural balance denoted by $X : YZ$ indicates, as usaul, if the variance of estimates of elementary treatment contrasts is constant or not. As the ultimate objective is analytical balance, the design classification notation should also be provided with a symbol indicating the position regarding analytical balance. Accordingly, a notation like $W(X:YZ)$ where W indicates the position of analytical balance is a more appropriate classificatory symbol. For example, a more appropriate notation for a Youden square design is $T\,(0 : T\,0)$.

Kshirsagar [1] has reported a row and column design in 9 treatments, 6 rows and 6 columns. The design is equireplicate, binary and is analytically balanced. Its appropriate classification is $T(0 : PP)$. Kshirsagar's design indicates that there may exist other balanced row-and-column binary designs having Y and Z components other than 0 or T. This is investigated in the present work.

2. A Property of Balanced Row and Column Designs

It is known (Pearce, [4]) that for a row and column design in p rows, q columns and v treatments such that each treatment is replicated r times, the reduced normal equations for estimating treatment effects adjusted for row and column effects, are

$$\underset{\sim}{F}\,\underline{t} = \underline{Q},$$

where

$$\underset{\sim}{F} = r\,\underset{\sim}{I} - \underset{\sim}{N_1}\,\underset{\sim}{N_1'}/q - \underset{\sim}{N_2}\,\underset{\sim}{N_2'}/p + \frac{r}{v}\,\underset{\sim}{J}, \tag{2.1}$$

$\underline{t}$ is the vector of treatment effects and

$$\underline{Q} = \underline{T} - \underset{\sim}{N_1'}\,\underline{R}/q - \underset{\sim}{N_2}\,\underline{C}/p + \underset{\sim}{N_1}\,\underline{1}\,G/pq$$

is the vector of adjusted treatment totals with $\underline{T}$, $\underline{R}$ and $\underline{C}$ being the vectors of treatment, row and column, $\underset{\sim}{I}$ is the identity matrix of order v, $\underline{1}$ is a vector of ones, G is the grand total and $\underset{\sim}{N_1}$ and $\underset{\sim}{N_2}$ are treatments verses rows and treatments verses columns incidence matrices respectively.

We write $\underset{\sim}{F}$ as

$$\underset{\sim}{F} = r\,\underset{\sim}{I} - \frac{1}{pq}\,\underset{\sim}{D} + \frac{r}{v}\,\underset{\sim}{J}, \tag{2.2}$$

where

$$\underset{\sim}{D} = p\,\underset{\sim}{N_1}\,\underset{\sim}{N_1'} + q\,\underset{\sim}{N_2}\,\underset{\sim}{N_2'}$$

The reduced normal equations $\underset{\sim}{F}\,\underset{\sim}{t} = \underset{\sim}{Q}$ will ensure analytial balance only if $\underset{\sim}{F}$ is a matrix with off-diagonal elements equal to a constant and the diagonal elements equal to another constant. Of the three components of $\underset{\sim}{F}$, namely, $\underset{\sim}{I}$, $\underset{\sim}{D}$ and $\underset{\sim}{J}$, the components $\underset{\sim}{I}$ and $\underset{\sim}{J}$ satisfy the above conditions. Though in the case of $\underset{\sim}{J}$, both the constants, off-diagonal and diagonal are equal, this does not matter as only for at least one component the two constants must be different. If $\underset{\sim}{D}$ also satisfies the requirements of constant off-diagonal and diagonal elements for a design, then the design will be analytically balanced.

The off-diagonal elements of $\underset{\sim}{N_1}\,\underset{\sim}{N_1'}$ are $\lambda_r(i, j)$, $(i \neq j = 1, \ldots, v)$, where $\lambda_r(i, j)$ is the number of rows each containing the ith and jth treatments together. Its diagonal elements are r each. Similarly, the off-diagonal elements of $\underset{\sim}{N_2}\,\underset{\sim}{N_2'}$ are $\lambda_{c(i,j)}$, where $\lambda_{c(i,j)}$ is the number of columns each containing the ith and jth treatments together. Its diagonal elements are also r each. Hence the off-diagonal elements of $\underset{\sim}{D}$ are $p\lambda_{r(i,j)} + q\lambda_{c(i,j)}$, $(i \neq j = 1, \ldots, v)$, and its diagonal elements are $r(p + q)$. Hence $\underset{\sim}{D}$ will ensure analytical balance if

$$p\lambda_{r(i,j)} + q\lambda_{c(i,j)} = \lambda, \tag{2.3}$$

where λ is a constant for all pairs of (i, j) and is given by

$$\lambda = r\,(2pq - p - q)/(v - 1)$$

This result is originally due to Singh [6].

3. Existence of Row and Column Balanced Designs

Singh (1980) tabulated parametric combinations of six possibly existent balanced designs in the range $3 \leqslant r \leqslant 10$; $r < p, q < v$ satisfying the condition (2.3), r being the number of replications of each treatment. These are given in Table 1.

TABLE 1

S. No.	v	r	p	q	λ
1	9	4	6	6	30
2	15	6	9	10	69
3	16	3	4	12	16
4	21	5	7	15	47
5	25	4	10	10	30
6	27	4	9	12	30

Design 1 was reported by Kshirsagar [1]. Design 4 can be deleted from the list as it violates the condition (2.3). The condition gives the equation

$$7\lambda_{r(i,j)} + 15\,\lambda_{c(i,j)} = 47$$

For no integral values of $\lambda_{r(i,j)}$ and $\lambda_{c(i,j)}$ the above equality holds.

For the designs 2 and 6, the condition (2.3) gives

$$9\lambda_{r(i,j)} + 10\,\lambda_{c(i,j)} = 69$$
$$9\lambda_{r(i,j)} + 12\,\lambda_{c(i,j)} = 30$$

giving $\lambda_{r(i,j)} = 1$, $\lambda_{c(i,j)} = 6$ for design 2 and giving $\lambda_{r(i,j)} = 2$, $\lambda_{c(i,j)} = 1$ for design 6. Since in each case there is only one value of pairwise occurence in rows as well as columns, it implies that there should exist a balanced incomplete block arrangement both row-wise and columnwise. Since p and q in each case is less than v no such arrangement exists. This proves the non existence of the designs 2 and 6.

An application of the condition (2.3) to design 3 gives the following two solutions

(i) $\lambda_{r(i,j)} = \lambda_{c(i,j)} = 1$, (ii) $\lambda_{r(i,j)} = 4$, $\lambda_{c(i,j)} = 0$.

Solution (ii) is not feasible as pairwise occurence in rows exceeds the numbers of replications. The row-and-column design is again non-existent as both p and q are less than v.

For design 5, we have the following equation

$$\lambda_{r(i,j)} + \lambda_{c(i,j)} = 3$$

This gives rise to the following feasible solutions :

(i) $\lambda_{r(i,j)} = 0$, $\lambda_{c(i,j)} = 3$
(ii) $\lambda_{r(i,j)} = 3$, $\lambda_{c(i,j)} = 0$
(iii) $\lambda_{r(i,j)} = 1$, $\lambda_{c(i,j)} = 2$
(iv) $\lambda_{r(i,j)} = 2$, $\lambda_{c(i,j)} = 1$

In remains to be investigated whether a solution to design 5 exists.

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